COMBINATORICA 7 (3) (1987) 303—312

SUBGRAPHS OF COLOUR-CRITICAL GRAPHS

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Received 7 May 1985

Some problems and results on the distribution of subgraphs in colour-critical graphs are discussed.

In section 3 arbitrarily large k-critical graphs with n vertices are constructed such that, in order to reduce the chromatic number to $k-2$, at least $c_k n^2$ edges must be removed.

In section 4 it is proved that a 4-critical graph with n vertices contains at most n triangles. Further it is proved that a k-critical graph which is not a complete graph contains a $(k-1)$-critical graph which is not a complete graph.

1. Introduction

Most of the concepts used in this paper can be found in [7, pp. 528—540]. All graphs considered are finite, undirected and have neither loops nor multiple edges.

A graph $G=(V, E)$ consists of a set $V=V(G)$ of vertices and a set $E=E(G)$ of edges. The number of vertices and the number of edges of G are denoted by $v(G)$ and $e(G)$, respectively.

For any graphs G_1 and G_2 put $G_1\cup G_2=(V(G_1)\cup V(G_2),\ E(G_1)\cup E(G_2))$ and $G_1\cap G_2=(V(G_1)\cap V(G_2),\ E(G_1)\cap E(G_2))$. If $F\subseteq E(G)$ then let $G-F=(V(G), E(G)-F)$. For an edge $e=\{P, Q\}\in E(G)$, G/e denotes the graph obtained from $G-\{e\}$ by identifying P and Q and replacing multiple edges by single ones.

K_n and C_n will denote the complete graph (also called complete n-graph) and the circuit on n vertices, respectively. The circuit C_n is called odd (even) if n is odd (even). $K_d(n_1, n_2, \ldots, n_d)$ $(d\geqq 2)$ denotes the complete d-partite graph with $n_1+\ldots+n_d$ vertices partitioned into the classes $X_1, \ldots, X_d$, $|X_i|=n_i$ $(i=1, \ldots, d)$, where two vertices are adjacent if and only if they belong to different classes. Let $T(n, d)=K_d(n_1, \ldots, n_d)$ with $n_i=\lfloor (n+i-1)/d\rfloor$ $(i=1, \ldots, d)$, where $\lfloor x\rfloor$ is the largest integer not greater than x.

AMS subject classification (1980): 05 C15, 05 C 35

A k-colouring of G is a mapping c of $V(G)$ into the set of integers $\{1, 2, \ldots, k\}$ such that $c(P) \neq c(Q)$ for any two adjacent vertices $P, Q \in V(G)$. The chromatic number $\chi(G)$ of a non-empty graph G is the smallest integer k for which G admits a k-colouring.

A graph G is called critical if $\chi(H) < \chi(G)$ for every proper subgraph H of G and it is called k-critical $(k \geqq 1)$ if it is critical and k-chromatic (i.e. $\chi(G)=k$).

Obviously, a connected graph G is critical if and only if $\chi(G-\{e\}) < \chi(G)$ for every edge e of G.

The important notion of critical graphs was introduced by G. A. Dirac [1], [2]; for major contributions to the theory of critical graphs see [3], [11]—[16].

A K_k is an example of a k-critical graph and for $k=1, 2$ it is the only one. The 3-critical graphs are the odd circuits. Let us mention two constructions of critical graphs:

The Dirac construction. Let G_1 and G_2 be two disjoint graphs with chromatic numbers k_1 and k_2, respectively. Let $G=G_1 \nabla G_2$ denote the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 by an edge. Then $\chi(G)=k_1+k_2$, and G is critical if and only if G_1 and G_2 are critical. This result is due to G. A. Dirac (see [3]).

The Hajós construction. Let G_1 and G_2 be two k-critical graphs $(k \geqq 3)$ such that the intersection graph $H=G_1 \cap G_2$ is a complete h-graph with $1 \leqq h \leqq k-2$. Further, for $i=1, 2$, let $e_i=\{P, Q_i\} \in E(G_i)$ where $P \in V(H)$, $Q_i \in V(G_i)-V(H)$ and in G_i the vertex Q_i is adjacent to all vertices of H. Put $e=\{Q_1, Q_2\}$. Then the graph G with $V(G)=V(G_1) \cup V(G_2)$ and $E(G)=(E(G_1) \cup E(G_2) \cup \{e\})-\{e_1, e_2\}$ is k-critical. This result was proved by G. Hajós [5] (for $h=1$), G. A. Dirac and T. Gallai (see [3, Satz (2.12)]).

If G_1 and G_2 are complete k-graphs $(k \geqq 3)$ and $H=G_1 \cap G_2 \cong K_1$ then the graph obtained by the above Hajós construction is denoted by L_k. Obviously, L_k is a k-critical graph with $2k-1$ vertices (note that $L_3 \cong C_5$). Figure 1.1 shows graph L_4.

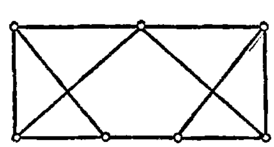

Fig. 1.1

The graph $G=C_n \nabla C_n$ ($n \geqq 3$ an odd integer) is, by the Dirac construction, a 6-critical graph with $e(G) \geqq \frac{1}{4} v(G)^2$.

In 1970 B. Toft [11] proved that for every $k \geqq 4$ there are arbitrarily large k-critical graphs G with many edges, i.e. with $e(G) \geqq c_k v(G)^2$ $(c_k > 0)$.

In section 2 it is proved that among the subgraphs of k-critical graphs ($k \geqq 4$) with n vertices (n large enough) there exists exactly one having the maximum number of edges, namely, $T(n, k-2)$.

Investigating critical graphs having many edges B. Toft was led to the question (see [14]) whether for a given integer $k \geqq 4$ there exist infinitely many k-critical graphs G and a positive constant c_k such that in order to reduce the chromatic number of G to $k-2$ at least $c_k v(G)^2$ edges must be removed from G. In section 3 we shall give an affirmative answer to this question.

Section 4 deals with critical subgraphs of critical graphs. J. Nešetřil and V. Rödl conjectured (see [14] or [16]) that a large k-critical graph G ($k \geqq 4$) contains a large ($k-1$)-critical subgraph H (i.e. if $v(G)$ tends to infinitely then $v(H)$ tends to infinity). This is true for $k=4$ (see [6] and [16]) and unsettled for $k \geqq 5$. In section 4 it is proved that a k-critical graph which is not a K_k contains a ($k-1$)-critical subgraph which is not a K_{k-1}. Further, we prove that a 4-critical graph with n vertices contains at most n triangles.

2. Forbidden subgraphs

Let G be a k-critical graph and $e=\{P, Q\}$ an arbitrary edge of G. Then, by definition, $G-\{e\}$ is ($k-1$)-colourable. Since G is not ($k-1$)-colourable, we conclude that $c(P)=c(Q)$ for every ($k-1$)-colouring c of $G-\{e\}$, hence $\chi(G/e) \leqq k-1$.

Therefore, if H is a subgraph of a k-critical graph then $\chi(H/e) \leqq k-1$ for every edge e of H. That the converse of this statement is also true was proved by D. Greenwell and L. Lovász [4].

Let us now construct two families of forbidden subgraphs of critical graphs.

The graph $W(l, d)$ is defined as $W(l, d)=C_l \triangledown K_d$ ($l \geqq 3$ and $d \geqq 1$) and it is called a d-wheel. A 1-wheel is briefly called a wheel. The d-wheel $W(l, d)$ is called odd (even) if l is odd (even).

By the Dirac construction, an odd d-wheel is a ($d+3$)-critical graph. An even d-wheel $W=C_{2l} \triangledown K_d$ is ($d+2$)-chromatic and if e is an edge of W which belongs to the circuit C_{2l} then $W/e \cong W(2l-1, d)$, i.e. $\chi(W/e)=d+3$, hence W is not contained in any ($d+3$)-critical graph. Thus we have obtained

Lemma 2.1. *If G is a k-critical graph ($k \geqq 4$) containing $W(l, k-3)$ as a subgraph, then $G \cong W(l, k-3)$ and l is an odd integer.* ▮

The graph $T^+(n, d)$ is obtained from $T(n, d)$ ($n > d \geqq 2$) by putting an additional edge into a class of $\lfloor (n+d-1)/d \rfloor$ vertices of $T(n, d)$. Note that $T(n, d)$ is d-chromatic but $T^+(n, d)$ is ($d+1$)-chromatic and contains a K_{d+1}.

Let $n \geqq 2d+1$. Then it is easy to see that there is an edge e in $T^+=T^+(n, d)$ such that T^+/e contains a K_{d+2}, hence $\chi(T^+/e) \geqq d+2$ (see figure 2.1 showing $T^+(7, 3)$).

Thus, no subgraph of a ($d+2$)-critical graph contains $T^+(n, d)$ ($n \geqq 2d+1$) as a subgraph. On the other hand, from the result of Greenwell and Lovász mentioned above we conclude that $T(n, d)$ can be imbedded into a ($d+2$)-critical graph.

In 1968 M. Simonovits [9] proved the following extremal result:

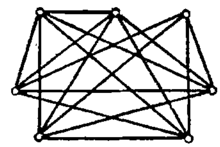

Fig. 2.1

Let H be a $(k+1)$-chromatic graph $(k \geqq 2)$ and assume that $H-\{e\}$ is k-colourable for some edge e of H. Then there is a positive integer n_0 such that for every $n \geqq n_0$, among all graphs with n vertices which do not contain H as a subgraph there is exactly one having the maximum number of edges, namely, $T(n, k)$.

As an easy consequence of this result and the above we obtain.

Theorem 2.2. *Let $k \geqq 4$ be an integer. Then there exists a positive integer $n_0 = n_0(k)$ such that the following statements hold.*

(1) *If H is a subgraph of a k-critical graph with $v(H) = n \geqq n_0$ vertices then $e(H) \leqq e(T(n, k-2))$, where equality holds if and only if $H \cong T(n, k-2)$.*

(2) *If G is a k-critical graph with $n \geqq n_0$ vertices then $e(G) < e(T(n, k-2))$.* ∎

3. A class of critical graphs

For a graph G and a positive integer k, let $m_k(G) = \min \{|F| \mid F \subseteqq E(G)$ and $\chi(G-F) \leqq k\}$. Note that $m_l(G) = e(G)$ and $m_l(G) = 0$ for $l \geqq \chi(G)$.

If G is k-critical then $m_{k-1}(G) = 1$ and one can ask how large $m_{k-2}(G)$ can be. Most of the known k-critical graphs G are "thin" in the sense that $m_{k-2}(G)/v(G)$ is very small.

By constructions of T. Gallai (see [8]), Z. Tuza and V. Rödl [15] it is known that there exist arbitrarily large k-critical graphs $G (k \geqq 4)$ with $m_2(G) = \binom{k-2}{2}$ (and this is best possible).

In this section we prove

Theorem 3.1. *Let $k \geqq 4$. Then there exist a positive constant c_k and arbitrarily large k-critical graphs G such that $m_{k-2}(G) \geqq c_k v(G)^2$.*

The proof will be given by means of an explicit construction similar to a construction of B. Toft (see [13]). The following result due to J. B. Kelly and L. M. Kelly will be very helpful.

Lemma 3.2 (J. B. Kelly and L. M. Kelly ([6.] *Let G be an odd circuit (i.e. G is a 3-critical graph) and let $f\colon V(G) \to \{1, 2, 3\}$ be a non-constant function. Then there exists a 3-colouring c of G such that $c(P) \neq f(P)$ for every $P \in V(G)$.* ∎

3.1. Construction

(A) Let $k \geqq 3$ be a given integer. The aim of our construction is a $(k+1)$-critical graph G_k. We start with the k-critical graph L_k introduced in section 1 (L_k is obtained from two complete k-graphs G_1, G_2 with $G_1 \cap G_2 \cong K_1$ by Hajós' construction). The proof of the following simple result is left to the reader.

Lemma 3.3. (a) *In any k-colouring of L_k one of the k colours occurs exactly once and all other colours occur exactly twice in L_k.*

(b) *Let P and Q be two non-adjacent vertices of L_k, $P \neq Q$. Then there exists a k-colouring c of L_k such that $c^{-1}(1)=\{P_1\}$, $c^{-1}(2)=\{P, Q\}$, $c^{-1}(3)=\{P', Q'\}$, and P_1 is not adjacent to P' in L_k.*

(c) *For every edge $e=\{P, Q\}$ of L_k there exists a k-colouring c of L_k satisfying $c(P)=2$, $c(Q)=3$ and $c^{-1}(1)=\{P_1\}$ where P_1 is neither adjacent to P nor to Q in L_k.* ∎

(B) For every $P \in V(L_k)$ fix a set $X(P)$ of at least two elements where the $X(P)$'s are pairwise disjoint. Put $M=\{X(P) | P \in V(L_k)\}$. Now define the graph $H_k=L_k(M)$ as follows:

$$V(H_k) = \cup M$$

$$E(H_k) = \big\{\{a, b\} \mid a \in X(P) \in M,\ b \in X(Q) \in M, \text{ and } \{P, Q\} \in E(L_k)\big\}.$$

Put $F_k=\big\{X(P) \cup X(Q) | P, Q \in V(L_k),\ P \neq Q,\ \{P, Q\} \notin E(L_k)\big\}$. Note that the sets of F_k are independent sets in H_k.

We say that a k-colouring c of $H_k=L_k(M)$ has the property U_3 with respect to $F' \subseteqq F_k$ if there exists a set $Y \in F'$ such that $c(Y)=\{\alpha\}$ for some $\alpha \in \{1, 2, 3\}$.

Next we shall prove the following crucial result.

Lemma 3.4. (1) *Every k-colouring of H_k has the property U_3 with respect to F_k.*

(2) *For every $Y \in F_k$ there is a k-colouring of H_k which does not have the property U_3 with respect to $F_k - \{Y\}$.*

(3) *For every edge e of H_k there is a k-colouring of $H_k - \{e\}$ which does not have the property U_3 with respect to F_k.*

Proof. Let c be an arbitrary k-colouring of H_k. It is easy to see that $|c(X)|=1$ for all but at most one set $X \in M$. Thus (1) is a simple consequence of Lemma 3.3(a).

Proof of (2). Let $Y=X(P) \cup X(Q) \in F_k$. Then P and Q are non-adjacent in L_k and there is a k-colouring c of L_k satisfying (b) of Lemma 3.3 (we choose the same notation). Now colour the vertices of $X(P')$ with 1 and 3 such that both colours occur and, for $R \in V(L_k) - \{P'\}$, assign colour $c(R)$ to all vertices of $X(R)$. Obviously, the k-colouring of H_k obtained this way does not have the property U_3 with respect to $F_k - \{Y\}$.

Proof of (3). Let $e=\{a, b\}$ be an edge of H_k where $a \in X(P)$ and $b \in X(Q)$. Let c be a k-colouring of L_k satisfying (c) of Lemma 3.3 with respect to the edge $\{P, Q\}$ of L_k (we choose the same notation). Give colour $c(R)$ to all the vertices of

$X(R)$, for $R\in V(L_k)$, except for the vertices a and b, which are both coloured with 1. This proves (3). ∎

(C) For $Y\in F_k$, let C_Y denote an odd circuit with at least $|Y|$ vertices. Further, let G' denote a $(k-3)$-critical graph (if $k=3$ then G' is empty) and let us assume that $H_k=L_k(M)$, G' $C_Y(Y\in F_k)$ are pairwise disjoint. Put $C=\{C_Y|Y\in F_k\}$. From the graphs $L_k(M)$, G', C_Y $(Y\in F_k)$ we obtain a graph $G_k=G(L_k(M), C, G')$ as follows: for every $Y\in F_k$, join each vertex of Y to one or more vertices of C_Y by an edge so that each vertex of C_Y is joined to precisely one vertex of Y.

Then join all vertices of C_Y, $Y\in F_k$, to all vertices of G'. Denote the graph obtained this way by G_k.

Lemma 3.5. *G_k is $(k+1)$-critical.*

Proof. We prove that G_k is not k-colourable, but $G_k-\{e\}$ is k-colourable for every edge e of G_k.

Suppose that G_k has a k-colouring, c. Since G' is $(k-3)$-critical, under c each of the odd circuits C_Y, $Y\in F_k$, has exactly the same three different colours, say 1, 2 and 3. But this contradicts Lemma 3.4 (see (1)). Hence $\chi(G_k)\geqq k+1$. It is easy to see that $G_k-\{e\}$ has a k-colouring for every e of G_k. Let us consider the most important case, where e is an edge of $H_k=L_k(M)$. By Lemma 3.4 (see (3)), there is a k-colouring c of $H_k-\{e\}$ which does not have the property U_3 with respect to F_k. Give colour $4, 5, \dots, k$ to the vertices of G'. Then the k-colouring c' of $H'=(H_k-\{e\})\cup G'$ obtained can be extended to a k-colouring of $G_k-\{e\}$, since every vertex of a circuit C_Y, $Y\in F_k$, is adjacent to at most one vertex P of H' with $c'(P)\in\{1, 2, 3\}$ (use Lemma 3.2). ∎

We are now ready to prove Theorem 3.1. For $n\geqq 2$ let G_k^n $(k\geqq 3)$ denote the graph $G(L_k(M), C, G')$ where $|X|=n$ for $X\in M$, $|V(C_Y)|=2n+1$ for $Y\in F_k$ and $G'\cong K_{k-3}$. Then G_k^n is a $(k+1)$-critical graph and there is a positive constant c_k such that $v(G_k^n)\leqq c_k n$ and $m_{k-1}(G_k^n)\geqq n^2$.

3.2. Some remarks

3.2.1. The fact that $L_3(M)$ is a subgraph of some 4-critical graph I learned from V. Rödl. Using different ideas V. Rödl obtained other examples of 4-critical graphs and proved the following stronger result (perconal communication): For every $k\geqq 2$ there exist a positive constant c_k and infinitely many 4-critical graphs G without odd circuits C_l of length $l\leqq 2k-1$ such that $m_2(G)\geqq c_k v(G)^2$.

3.2.2. B. Toft drew my attention to the fact that the above examples of 4-critical graphs not only solve his problems but also provide a best possible negative answer to a question of T. Gallai, who asked (see [11]) whether Lemma 3.2 can be extended to 4-critical graphs (B. Sørensen and B. Toft proved that such a result does not hold for k-critical graphs with $k\geqq 5$, see [13]).

Theorem 3.6. *There exists a 4-critical graph G and a non constant function f: $V(G)\to\{1, 2, 3\}$ such that for any 4-colouring c of G there is always a vertex P of G with $c(P)=f(P)$.*

Proof. Let $G=G_3^3=G(L_3(M), C, G')$ be the 4-critical graph of the above construction, where each of the five classes of M consists of precisely three elements. Choose f so that f takes all values 1, 2, 3 on each of the five classes of M. Clearly, this proves Theorem 3.6 (note that $L_3 \cong C_5$). ▮

3.2.3. Let S_k denote the set of all proper subgraphs of k-critical graphs. In connection with the above construction the author was led to the following.

Conjecture 3.7. For every $k \geqq 4$ there exists a positive constant c_k such that for every $H \in S_k$ there is a k-critical graph G with at most $c_k v(H)$ vertices containing H as a subgraph.

Let us mention the following partial results.

Theorem 3.8. (B. Toft [13]). *For every $k \geqq 4$ there exists a constant M_k such that every $(k-2)$-colourable graph H is contained in a k-critical graph G of at most $2v(H)+M_k$ vertices.* ▮

Theorem 3.9. (H. Sachs and M. Stiebitz [9]). *Let $k \geqq 4$ and $\alpha \geqq 1$. Then there exists a constant $c=c_{k,\alpha}$ such that any connected graph $H \in S_k$ is contained as an induced subgraph in a k-critical graph G having the following properties.*

(a) *In G every vertex of H has degree $\geqq \alpha$,*
(b) *every vertex of $G-H$ has degree $k-1$, and*
(c) $v(G) \leqq ce(H) \leqq cv(H)^2$. ▮

4. Critical subgraphs of critical graphs

The following two problems are due to T. Gallai (oral communication).

Problem 4.1. Let G be a k-critical graph $(k \geqq 4)$ on n vertices.

(4.1.1) Is it true that the number of $(k-1)$-critical subgraphs of G is $\geqq n$?

(4.1.2) Is it true that the number of complete $(k-1)$-graphs contained in G is $\leqq n$ with equality if and only if $G \cong K_k$?

In 1974 B. Toft proved.

Theorem 4.2. (B. Toft [12]). *Let G be a k-critical graph $(k \geqq 3)$ and let e_1 and e_2 be any two edges of G. Then there is a $(k-1)$-critical subgraph of G containing e_1, but not containing e_2.* ▮

Let $H_1, H_2, \ldots, H_l$ be all $(k-1)$-critical subgraphs of a k-critical graph G. Further, for an edge e of G, let $A(e)$ denote the set of all those integers i $(1 \leqq i \leqq l)$ for which e belongs to H_i. Then $A(e) \subseteqq \{1, 2, \ldots, l)$ and, by Theorem 4.2, none of these sets contains another one. Therefore,

$$v(G) \leqq e(G) \leqq \binom{l}{\lfloor l/2 \rfloor} \leqq 2^l$$

and we obtain

Theorem 4.3. *A k-critical graph* $(k \geqq 4)$ *on n vertices containes at least* $\log_2 n$ *subgraphs which are* $(k-1)$*-critical.* ▮

Next we prove

Theorem 4.4. *Let G be a* 4*-critical graph on n vertices. Then the number of triangles in G is* $\leqq n$.

The proof uses linear algebra. Some further notation is needed.

Let G be a graph and let $V(G)=\{P_1, P_2, \ldots, P_n\}$. Further, let $T_k(G)=\{H|H$ is a subgraph of G and $H \cong K_k\}$ and define, for every graph $H \in T_k(G)$, a vector $v_H=(v_1, \ldots, v_n)^T$ where

$$v_i = \begin{cases} 1 & \text{if} \quad P_i \in V(H) \\ 0 & \text{otherwise.} \end{cases}$$

Put $t_k(G)=|T_k(G)|$. Instead of Theorem 4.4 we prove the following stronger result.

Theorem 4.4′. *Let G be a* 4*-critical graph. Then the vectors* $v_H(H \in T_3(G))$ *are linearly independent over* GF(2).

Proof. If G is a K_4, then this is obvious. Therefore, in what follows, let us assume that G is not a K_4. For $e \in E(G)$ or $P \in V(G)$, let $d(e\colon T')$ and $d(P\colon T')$ denote the number of all graphs from $T' \subseteqq T_3(G)$ containing e or P, respectively. Now we prove the following two statements from which Theorem 4.4′ is an easy consequence.

Let $T' \subseteqq T_3(G)$, $T' \neq \emptyset$. Then

(A) there is an edge e of G such that $d(e\colon T') \equiv 1 \pmod 2$, and
(B) there is a vertex P of G such that $d(P\colon T') \equiv 1 \pmod 2$.

Proof of (A). Assume that $d(e\colon T')$ is even for every edge e of G. Let P be a vertex of G which belongs to some graph of T', and let $H_1, H_2, \ldots, H_r \in T'$ be all those graphs containing P. Put $G'=H_1 \cup H_2 \cup \ldots \cup H_r$ and $G''=G'-\{P\}$. From the assumption we conclude that all vertices of G'' have even degrees in G'' and therefore, G'' contains a circuit. Since P is adjacent to all vertices of G'' it follows that $G' \subseteqq G$ contains a wheel. Then, by Lemma 2.1, G is an odd wheel and, clearly, (A) holds (note that $G \not\cong K_4$). ▮

Proof of (B). Because of (A), there is an edge in G, say $e=\{P_1, P_2\}$, such that $d(e\colon T') \equiv 1 \bmod 2$. Since G is 4-critical, there exists a 3-colouring c of $G-\{e\}$ with $c(P_1)=c(P_2)=1$. Let $X=\{P|P \in V(G)$ and $c(P) \in \{2, 3\}\}$: then, for $H \in T'$

$$|V(H) \cap X| = \begin{cases} 1, & \text{if} \quad e \in E(H); \\ 2, & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{P \in X} d(P\colon T') = d(e\colon T') + 2(|T'| - d(e\colon T'))$$

and, because $d(e\colon T')$ is odd, there is a vertex $P \in X$ for which $d(P\colon T')$ is odd. ▮

Remark 4.5. Let G be a k-critical graph ($k \geqq 5$). Further, let $T' \subseteq T_{k-1}(G)$ and suppose that G is not a K_k. By a similar argument as in the proof of (A), it can be shown that there exists a set of $k-3$ vertices of G contained in an odd number of graphs from T'. Therefore, we obtain $t_{k-1}(G) \leqq \binom{n}{k-3}$ where $n = v(G)$.

On the other hand we can prove

Theorem 4.6. *For every $k \geqq 4$ there exists a $c_k > 0$ such that there are arbitrarily large k-critical graphs G with $v(G) = n$ and $t_l(G) \geqq c_k n^l$ for $l = 2, 3, \ldots, k-2$.*

Proof. Let $n \geqq 2$ and $H_n = K_l(n, n, \ldots, n)$ $(2 \leqq l \leqq k-2)$. Then, by Theorem 3.8, there is a k-critical graph G containing H_n as a subgraph for which $v(G) \leqq c'n$. Clearly, $t_l(G) \geqq n^l$. ▮

The following result was conjectured by J. Nešetřil and B. Toft (oral communication). Note that this result would be a simple consequence of an affirmative answer to Problems 4.1.1 and 4.1.2.

Theorem 4.7. *If G is a k-critical graph $(k \geqq 4)$ all of whose $(k-1)$-critical subgraphs are isomorphic to K_{k-1}, then G is isomorphic to K_k.*

Proof. Clearly, G contains a $(k-1)$-critical subgraph and therefore, there is a K_3 in G. Let e_1, e_2 and e_3 be three edges of G forming a K_3. From Theorem 4.2 we conclude that G contains two $(k-1)$-critical subgraphs, say H_1, H_2, where H_i contains e_i but does not contain e_3 $(i=1, 2)$. By the assumption, H_1 and H_2 are both complete $(k-1)$-graphs, thus $e_2 \notin E(H_1)$, $e_1 \notin E(H_2)$ and $H = H_1 \cap H_2$ is a complete graph, say, on h vertices where $1 \leqq h \leqq k-2$. If $h = k-2$ then the graph G' obtained from $H_1 \cup H_2$ by adding the edge e_3 is a complete k-graph which is contained in G and therefore, $G \cong K_k$. If $h < k-2$ then, by the Hajós construction, $G' = (V(H_1) \cup \cup V(H_2), (E(H_1) \cup E(H_2) \cup \{e_3\}) - \{e_1, e_2\})$ is a $(k-1)$-critical subgraph of G which is not isomorphic to K_{k-1}, contradicting the assumption. ▮

As a simple consequence, we obtain

Theorem 4.8. *Let G be a k-critical graph $(k \geqq 4)$ which is not a complete k-graph. Then, for every h $(3 \leqq h \leqq k-1)$, there exists a h-critical subgraph of G which is not a complete h-graph.* ▮

Investigating critical subgraphs of critical graphs the author was led to the following.

Conjecture 4.9. Let G be a k-critical graph $(k \geqq 4)$ which is not a complete k-graph. Then there exists a $(k-1)$-critical subgraph of G which is not an induced subgraph of G.

References

[1] G. A. DIRAC, A property of 4-chromatic graphs, *J. London Math. Soc.* **27** (1952), 85—92.
[2] G. A. DIRAC, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69—81.
[3] T. GALLAI, Kritische Graphen I, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963), 165—192.
[4] D. GREENWELL and L. LOVÁSZ, Applications of product colouring, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 335—340.
[5] G. HAJÓS, Über eine Konstruktion nicht n-färbbarer Graphen, *Wiss. Z. Martin Luther Univ. Halle-Wittenberg Math. Nat.* *X*/1 (1961), 116—117.
[6] J. B. KELLY and L. M. KELLY, Paths and circuits in critical graphs, *Amer. J. Math.* **76** (1954), 786—792.
[7] L. LOVÁSZ, *Combinatorial Problems and Exercises*, Akad. Kiadó, Budapest 1979.
[8] F. NIELSEN and B. TOFT, On a class of 4-chromatic graphs due to T. Gallai, *in: Recent advances in Graph Theory, Proceedings of the Symposium held in Prague,* June 1974, Acad. Prague 1975, 425—430.
[9] H. SACHS and M. STIEBITZ, Construction of colour-critical graphs with given major-vertex subgraphs, *Annals of Discr. Math.* **17** (1983), 581—598.
[10] M. SIMONOVITS, A method for solving extremal problems in graph theory, stability problems, *in: Theory of Graphs, Proc. Colloq. Tihany, 1966,* Academic Press, New York, 1968.
[11] B. TOFT, On the maximal number of edges of critical k-chromatic graphs, *Studia Sci. Math. Hung.* **5** (1970), 461—470.
[12] B. TOFT, On critical subgraphs of colour-critical graphs, *Discrete Math.* **7** (1974), 377—392.
[13] B. TOFT, Color-critical graphs and hypergraphs, *J. Comb. Theory Ser. B* **16** (1974), 145—161.
[14] B. TOFT, Graph colouring: A survey of some problems and results, *in: Proc. of the Second Danish-Polish Math. Programming Seminar DAPS 79,* DIKU Univ. of Copenhagen, Report. No **80**, April 1980, 351—363.
[15] Z. TUZA and V. RÖDL, On colour critical graphs, *J. Comb. Th. B,* **38** (1985), 204—213.
[16] H. J. VOSS, Graphs with prescribed maximal subgraphs and critical chromatic graphs, *Comment. Math. Univ. Carol.* **18** (1977), 129—142.

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